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The generalized Casimir operator and tensor representations of groups

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Abstract

There has been proposed a new method of the constructing of the basic functions for spaces of tensor representations of the Lie groups with the help of the generalized Casimir operator. In the definition of the operator there were used the Lie derivatives instead of the corresponding infenitisemal operators. When introducing the generalized Casimir operator we use the metric for which a group being considered will be isometry that follows from the invariance condition for the generalized Casimir operator. This allows us to formulate the eigenvalue and eigenfunction problems correctly. The invariant projection operators have been constructed in order to separate irreducible components. The cases of the Bianchi type G^3IX and G^3II groups are considered as examples.

I. INTRODUCTION

One of the ways to construct basic functions for the space of representation of the group G is to solve the eigenvalue problem for some complete set of the invariant commutative Casimir operators. [1,2]. These operators are defined in the space of scalar functions. Note, that the Casimir operators defined in a usual way in the space of functions with tensor values are not invariant ones. To construct basic functions for the space of the tensor representation of the group G, one decomposes objects, being considered, into irreducible components, which, in their turn, are not the eigenfunctions of the Casimir operator. Thus basic functions for the rotation group $SO(3)$ are the eigenfunctions of the Laplacian on two-sphere S^2, although the irreducible components of the tensors are eigenfunctions of the Laplacian on the group $SU(2)$ [3].

In the work [4] there was introduced the generalized Casimir operator G, which is invariant in the space of tensor functions. This made it possible to formulate correctly and solve

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the eigenvalue and tensor eigenfunction problems, and, thereby, to construct tensor representations of various groups. However, in this approach, the system of differential equations appears with "tangled" components of tensor functions. That is why the direct solving of the tensor eigenfunction problem, generally speaking, is impossible. Moreover, due to the decomposibility of the tensor representation into irreducible ones, the spectrum of the generalized Casimir operator is, in the general case, degenerate. Thus it is necessary to decompose invariantly and to disentangle the mentioned system. It means that we need to go over to the set of differential equations for individual components or for their combinations. This reduces, first, to the separating of the irreducible combinations of the components for a given tensor field and to the classification of these combinations [5], and, second, to the decomposition of the generalized Casimir operator into irreducible parts.

On the other hand in a number of works [6] - [9], the authors constructed the theory of split-structures $\mathcal{H}^r$ on the pseudo-Riemannian manifold M^n and considered its applications. The theory of split structures is a general approach to the decomposition of the tangent bundle of pseudo-Riemannian manifolds into r subbundles and the associated decomposition of geometric objects. There have been considered split-structures induced by groups of isometries and introduced the notion of the split-structure compatible with a given group of isometry. It turns out that the theory of split-structures on manifolds give us an accurate and natural technique for solving of the problems arising when finding the tensor eigenfunctions. The technique makes it possible to "disentangle" the corresponding system of differential equations, and, in fact, give us the method for invariant decomposition of the generalized Casimir operator into irreducible parts. In other words, the method separates variables in a tensor differential equation in partial derivatives generated by a given generalized Casimir operator when dealing with the tensor eigenfunction problem.

In the present work we consider applications of the notions and methods developed in [4] - [9] to the construction of tensor representations for groups acting on a given manifold M^n as groups of isometries.

The method can be applied in physics to the invariant defining and constructing of tensor multipoles (tensor harmonics) as well as to the obtaining of expansions of tensor physical fields in terms of multipoles, in fact, for any continuous groups. This finds its application to the classification and computing of all linear perturbations of a gravitational field for spaces of General Relativity with determined symmetries.

The work is organized in the following way. In Sec.II one defines the generalized Casimir operator. The latter differs from the usual Casimir operator. In the definition of the generalized Casimir operator the Lie derivatives with respect to the infinitesimal generators are used instead of the infinitesimal generators of a group. These generators are the tangent vectors to the one-parameter subgroup, or, in other words, are curves in M^n. The invariance condition of the generalized Casimir operator yields the equation for the metric which is used in this operator's definition. Hence it follows that the vectors being considered will be the Killing's vectors for some metric. Then we formulate the eigenvalue and tensor eigenfunction problems for the generalized Casimir operator.

In Sec.III the problem of dioganalization of the generalized Casimir operator is solved by using of a split-structure $\mathcal{H}^r$ compatible with the group G^r. The decomposition of the corresponding system of differential equations for tensor eigenfunctions into invariant irreducible differential subsystems is constructed in order to disentangle the system. For the

separating of the irreducible components of tensor eigenfunctions we use the technique of the invariant projection operators. As a result, the generalized Casimir operator splits into the set of the invariant operators for each irreducible component.

In Sec.IV the generalized Casimir operator are constructed for the rotation group $SO(3)$ in the three-dimensional Euclidean space. The construction of the spherical tensor symmetric harmonics of type $(2,0)$ and weight l is given as an instance.

In Sec.V we deal with nonunitary representations for noncompact groups $G^3\ II$ according to the Bianchi classification. There is constructed the generalized Casimir operator and given an example of the construction of a point series of the representation.

II. THE GENERALIZED CASIMIR OPERATOR AND TENSOR REPRESENTATIONS OF GROUPS

Let M^n be an n-dimensional manifold and G^r be an r-parameter transformation group on M^n. For any differentiable vector fields X, Y, Z, ξ, e_a on M^n the Lie bracket $[X,Y] = -[Y,X]$ obeys the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{2.1}$$

The Lie derivatives of a function φ, vector field Y, and one-form ω with respect to a vector X are given by the formulas

$$\mathcal{L}_X\varphi = X\varphi; \qquad L_X Y = [X,Y]; \qquad (L_X\omega)\cdot Z = X(\omega\cdot Z) - \omega\cdot[X,Z], \tag{2.2}$$

where $\omega\cdot Z = \omega(Z)$ is an inner product of a one-form ω and vector Z. The Lie derivative $L_X\ T$ of a tensor T of type (p,q) with respect to a vector X is given by

$$(L_X\ T)(Y_1,...,Y_q) = L_X\ (T(Y_1,...,Y_q)) - \sum_{i=1}^{q} T(Y_1,...,Y_{i-1}, L_X\ Y_i, Y_{i+1},...,Y_q). \tag{2.3}$$

The Lie algebra of the group G^r is represented by the vector fields ξ_i $(i = 1,2,...,r)$, which are tangent to the one-parameter subgroups and have the properties

$$[\xi_i, \xi_j] = C^k_{ij}\xi_k. \tag{2.4}$$

Here C^k_{ij} are the structure constants satisfying the condition

$$C^k_{ij} = -C^k_{ji}; \qquad C^p_{is}C^s_{jk} + C^p_{js}C^s_{ki} + C^p_{ks}C^s_{lj} = 0. \tag{2.5}$$

Using the Jacobi identity we can easily find the relations for the commutators of the Lie derivatives.

$$[L_{\xi_i}, L_{\xi_k}] = L_{[\xi_i,\xi_k]} = C^j_{ik}L_{\xi_j}; \tag{2.6}$$

$$[L_{\xi_i},[L_{\xi_j},L_{\xi_k}]] + [L_{\xi_j},[L_{\xi_k},L_{\xi_i}]] + [L_{\xi_k},[L_{\xi_i},L_{\xi_j}]] = 0. \tag{2.7}$$

Thus the operators $L_{\xi_i},\ i = 1,2,...r$ form the representation of the Lie algebra for the group G^r. However, unlike the Lie algebra of the tangent vectors ξ_i, defined only on functions,

the Lie derivatives, defined on tensors of an arbitrary type and rang, are invariant operators under the general coordinate transformations. Hence we obtain the way of construction of the generalized Casimir operator of the second order

$$G = g^{ik} L_{\xi_i} L_{\xi_k}, \tag{2.8}$$

where g^{ik} $(i, k = 1, 2, ...r)$ are contravariant components of some unknown metric, which is subject to be determined. By its definition this operator commutes with all the operators L_{ξ_i} of the representation. Hence it follows

$$(L_{\xi_j} g^{-1})^{ik} \equiv \xi_j g^{ik} + C^i_{jl} g^{lk} + C^k_{jl} g^{il} = 0. \tag{2.9}$$

Here the tensor

$$g^{-1} = g^{ik} \xi_i \otimes \xi_k \qquad (rang\, ||g^{-1}|| = s) \tag{2.10}$$

defines the metric on covectors which belong to the surfaces of transitivity $M^s \subset M^n (s \leq r,\ s \leq n)$,where the symbol $\otimes$ denotes the tensor product. In accordance with (2.9), the group G^r on the surfaces of transitivity is a group of isometries, where ξ_i are the Killing's vectors. Solutions of the Killing's equations (2.9) give us the metric for the generalized Casimir operator (2.8).

It turns out that for semisimple groups that will be enough to consider only constant solutions of (2.9). Indeed, in this case the Cartan tensor

$$g_{ik} = \frac{1}{2} C^l_{ij} C^j_{lk}, \tag{2.11}$$

satisfying the equation

$$C^l_{ij} g_{lk} + C^l_{ik} g_{jl} = 0 \tag{2.12}$$

is non-degenerate [2]. Therefore there is the inverse of the tensor g^{ik} : $(g^{ik} g_{kj} = \delta^i_j)$, for which the conditions (2.12) yield

$$C^i_{jl} g^{lk} + C^k_{jl} g^{il} = 0. \tag{2.13}$$

Comparing (2.13) and (2.9) we can conclude that $\xi_j g^{ik} = 0$ and g^{ik} are constants on the surfaces of transitivity.

In the general case the consideration of the constant solutions of (2.9) will be inadequate for construction of the generalized Casimir operator G, which is non-degenerate on M^s.

Let T be a tensor of type (p, q) on M^s. Then we have

$$T = T^{a_1 \dots a_p}_{b_1 \dots b_q} e_{a_1} \bigotimes \dots \bigotimes e_{a_p} \bigotimes e^{b_1} \bigotimes \dots \bigotimes e^{b_q}, \tag{2.14}$$

where $e_a (a = 1, 2, ..., s)$ is some vector basis on M^s, and e^a, $e^a(e_b) = \delta^a_b$, is a co-vector basis on M^s. Here $T^{a_1 \dots a_p}_{b_1 \dots b_q}$ are the components of the tensor T with respect to the basis e_a. It will be the eigenfunction tensor of the generalized Casimir operator G, provided the equation

$$GT \equiv g^{ik} L_{\xi_i} L_{\xi_k} T = \lambda T. \tag{2.15}$$

is satisfied. This equation can be rewritten in the following form

$$\mathcal{G}T^{a_1\ldots a_p}_{b_1\ldots b_q} \equiv g^{ik}\mathcal{L}_{\xi_i}\mathcal{L}_{\xi_k}T^{a_1\ldots a_p}_{b_1\ldots b_q} = \lambda T^{a_1\ldots a_p}_{b_1\ldots b_q}, \tag{2.16}$$

where $\mathcal{G}T^{a_1\ldots a_p}_{b_1\ldots b_q} \equiv (GT)^{a_1\ldots a_p}_{b_1\ldots b_q}$ are representations of the generalized Casimir operator acting on the tensor T in the basis e_a. The representation of the Lie derivative is determined by the formula

$$\mathcal{L}_{\xi_i}T^{a_1\ldots a_p}_{b_1\ldots b_q} \equiv (L_{\xi_i}T)^{a_1\ldots a_p}_{b_1\ldots b_q} = T^{a_1\ldots a_p}_{b_1\ldots b_q,c}\xi^c_i + T^{a_1\ldots a_p}_{c,b_2\ldots b_q}\xi^c_{i,b_1} + \cdots$$

$$+ T^{a_1\ldots a_p}_{b_1\ldots b_{q-1}c}\xi^c_{i,b_q} - T^{ca_2\ldots a_p}_{b_1\ldots b_q}\xi^{a_1}_{i,c} - \cdots - T^{a_1\ldots a_{p-1}c}_{b_1\ldots b_q}\xi^{a_p}_{i,c}. \tag{2.17}$$

It is difficult to solve the equation (2.16) directly, because the Lie operators "tangle" components of T.

III. A SPLIT-STRUCTURE ON A MANIFOLD AND DIOGONALIZATION OF THE GENERALIZED CASIMIR OPERATOR

In order to solve the system of equations (2.16) it is necessary to "disentangle" components of the tensor T in the equations, i.e. to diogonalize the operator G. By means of decomposition the technique of the dioganalization of the generalized Casimir operator G can be realized invariantly. Herewith the tensor equations (2.16) split into the system of scalar differential equations for irreducible components of the tensor T. Now some additional definitions we used will be given below [8]- [9] .

A linear operator L on the tangent bundle $T(M)$ is a tensor of type $(1,1)$ which acts according to the relation $L\cdot X \equiv L(X) \in T(M),\ \ \forall X \in T(M)$. Then the formula

$$(L^T\cdot\omega)(X) = (\omega\cdot L)(X) \equiv \omega(L(X)), \qquad \forall X \in T(M) \tag{3.1}$$

determines L^T, a transpose of an operator L, which acts on a one-form ω.

The product of two linear operators $L\cdot H$ is defined by $(L\cdot H)\cdot X = L\cdot(H\cdot X) \in T(M),\ \forall X \in T(M)$. An operator H is called a symmetric one if

$$(H\cdot X, Y) = (X, H\cdot Y),\ \forall X, Y \in T(M)$$

.

We shall say that **a split structure** $\mathcal{H}^s$ is introduced on M if the s linear symmetric operators $H^a(a=1,2,\ldots s)$ of a constant rank with the properties

$$H^a\cdot H^b = \delta^{ab}H^b; \qquad \sum_{a=1}^{s}H^a = I, \tag{3.2}$$

where I is the unit operator $(I\cdot X = I,\ \ \forall X \in T(M))$, are defined on $T(M)$.

Then we can obtain the decomposition the tangent bundle $T(M)$ and cotangent bundle $T^*(M)$ into the $(n_1+n_2+\ldots+n_s)$ subbundles Σ^a, Σ^*_a, so that

$$T(M) = \bigoplus_{a=1}^{s}\Sigma^a; \quad T^*(M) = \bigoplus_{a=1}^{s}\Sigma^*_a\,. \tag{3.3}$$

Then arbitrary vectors, covectors, and metrics are decomposed according to the scheme:

$$X = \sum_{a=1}^{s} X^a, \quad \omega = \sum_{a=1}^{s} \omega_a, \quad g = \sum_{a=1}^{s} g^a, \quad g^{-1} = \sum_{a=1}^{s} g_a^{-1} \tag{3.4}$$

where

$$X^a = H^a \cdot X, \quad H^b \cdot X^a = 0, \quad X^a \cdot X^b = 0, \quad (a \neq b) \tag{3.5}$$

$$\omega_a = \omega \cdot H^a, \quad \omega_a(X^b) = 0, \quad (a \neq b). \tag{3.6}$$

Using this scheme we can obtain the decomposition of more complex tensors.

Let us now introduce an auxiliary definition. We shall say that a split structure $\mathcal{H}^s$ is compatible with a group of isometries if the conditions of invariance of $\mathcal{H}^s$ are satisfied, i.e. if

$$L_{\xi_i} H^a = 0, \qquad (i = 1, 2, ...r; a = 1, 2, ...s). \tag{3.7}$$

The equations (3.2), (3.7) define the invariant projection tensors. The integrability conditions of (3.7) are satisfied for solutions of (3.7) owing to (3.2).

In order to construct the projectors we require that there exist such dual vector $\{e_a\}$ and covector $\{e^b\}$ bases on M^s, that

$$e_a \cdot e^b = \delta_a^b\ ; \quad H^a = e_a \otimes e^a. \tag{3.8}$$

From now on we shall not sum on repeating indices a and b. The invariance condition of (3.7) yields

$$(L_{\xi_i} e^a) \cdot e_b = 0 \qquad (a \neq b). \tag{3.9}$$

Hence it follows

$$L_{\xi_i} e^a = \mu_i^a e^a, \tag{3.10}$$

where the factors of proportionality μ_i^a are some functions, satisfying the equation

$$\xi_i \mu_k^a - \xi_k \mu_i^a = C_{ik}^j \mu_j^a, \tag{3.11}$$

which follows from the integrability condition of (3.10). Using (2.2) and (3.10) we find

$$L_{\xi_i} e_a = -\mu_i^a e_a. \tag{3.12}$$

Thus, the problem of the construction of the invariant projectors reduces to the construction of the dual vector $\{e_a\}$ and covector $\{e^b\}$ bases satisfying the system of equations (3.10)-(3.12). Some of the factors μ_i^a, or even all of them in some cases, can vanish. Then the projectors are constructed by means of the invariant basis $\{e_a : L_{\xi_i} e_a = 0\}$. Thus in the case of a simply transitive group ($r = s$), the invariant vector basis $\{e_a\}$ can be expressed in the form

$$e_a = L^b_a \xi_b \qquad (det||L^b_a|| \neq 0). \tag{3.13}$$

The factors L^b_a satisfy the equations

$$\xi_b L^a_d + C^a_{bq} L^q_d = 0. \tag{3.14}$$

The integrability conditions of these equations are satisfied owing to the Jacobi identity.

If the invariant vector basis $\{e_a\}$ on M^s is determined then the inverse metric (2.10) in the case of the Riemannian manifolds can be constructed by the formulas

$$g^{-1} = \delta^{ab} e_a \otimes e_b = g^{ab} \xi_a \otimes \xi_b \; ; \qquad g^{ab} = L^a_c L^b_d \delta^{cd}. \tag{3.15}$$

Using (3.14) it easily can be seen that the tensor g^{-1}, constructed in accordance with (3.15), actually satisfies the Killing's equations (2.9).

In any case of bases $\{e_a, e^b\}$ the initial tensor T can be expanded in the series

$$T = \sum_{A,B} \hat{T}^A_B = \sum_{A,B} T^A_B \hat{e}^B_A, \tag{3.16}$$

where $\{\hat{e}^B_A\} = \{e_{a_1} \otimes \cdots \otimes e_{a_p} \otimes e^{b_1} \otimes \cdots \otimes e^{b_q}\}$ is the tensor basis, $\hat{T}^A_B = T^A_B e^B_A$ is the tensor monomial and $T^A_B \equiv T^{a_1 \ldots a_p}_{b_1 \ldots b_q}$ is its component. $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$ are collective indices. The sum in (3.16) comprises the complete set of indices A, B. It is easy to show that since the projectors H_a are invariant, the eigenvalue equations (2.15) and (2.16) split into the set of independent eigenvalue invariant equations for monomials

$$G\hat{T}^A_B \equiv g^{ik} L_{\xi_i} L_{\xi_k} \hat{T}^A_B = \lambda \hat{T}^A_B.$$

Using this relation together with (3.10) and (3.11) we obtain

$$\mathcal{G} T^A_B \equiv g^{ik} \mathcal{L}_{\xi_i} \mathcal{L}_{\xi_k} T^A_B = g^{ik} (\xi_i - \phi^A_{iB})(\xi_k - \phi^A_{kB}) T^A_B = \lambda T^A_B. \tag{3.17}$$

Here

$$\phi^A_{iB} = \sum_{k=1}^{p} \mu_i^{a_k} - \sum_{n=1}^{q} \mu_i^{b_n}, \quad A = \{a_1, \ldots, a_p\}, \quad B = \{b_1, \ldots, b_q\}. \tag{3.18}$$

Thus in order that the tensor equation (2.15) could go over into the invariantly split equations (3.17) for the irreducible components T^A_B, we must make a change

$$T \to T^A_B; \qquad L_{\xi_i} \to \mathcal{L}_{\xi_i} = \xi_i - \phi^A_{iB}.$$

The equations (3.17) can be rewritten in the form

$$\mathcal{G} T^A_B = [K - 2g^{ik} \phi^A_{iB} \xi_k - g^{ik} \xi_i \phi^A_{kB} + g^{ik} \phi^A_{iB} \phi^A_{kB}] T^A_B = \lambda T^A_B, \tag{3.19}$$

where

$$K = g^{ik} \xi_i \xi_k \tag{3.20}$$

is the standard Casimir operator defined in the space of scalar functions. The solutions of the equations (3.19) and (3.10), (3.11) give us the basic tensor functions $\hat{T^A_B} = T^A_B \hat{e^B_A}$ in the space of a tensor representation of the group G^r (or, in other words, tensor harmonics). Note that if there is the invariant basis (3.13), then the generalized Casimir operator (2.15) with respect to this basis reduces to the standard Casimir operator K, and in order to construct the tensor basis of representation that will be enough to determine the basis of representation in the space of scalar functions.

IV. THE GENERALIZED CASIMIR OPERATOR FOR THE ROTATION GROUP AND ITS TENSOR REPRESENTATIONS

Let us consider, as an example and comparison with the known results, the case of the rotation group $SO(3)$ in the three-dimensional Euclidean space. The Lie algebra of this group in terms of the spherical coordinate system is represented by the following tangent vectors [10]:

$$\xi_1 = \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\,; \quad \xi_2 = -\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\,; \quad \xi_3 = -\frac{\partial}{\partial\varphi}. \tag{4.1}$$

The Lie bracket is $[\xi_i, \xi_j] = \varepsilon_{ijk}\xi_k$, where ε_{ijk} are the Levi-Civita's symbols. As a usual one should go over into the vectors generating the creation and annihilation operators.

$$H_s = -s\xi_2 + \imath\xi_1 = e^{\imath s\varphi}\left(s\frac{\partial}{\partial\theta} + \imath\cot\,\theta\frac{\partial}{\partial\varphi}\right); \qquad H_3 = -\imath\frac{\partial}{\partial\varphi}; \qquad s = \pm 1. \tag{4.2}$$

In the case being considered the surfaces of transitivity $M^2 = S^2 : r = const$ are two-dimensional, although the space of the group is three-dimensional. The Cartan tensor (2.11) is non-degenerate in this case, and $g_{ik} = g^{ik} = \delta_{ik}$. Therefore we can take the operator

$$K = \xi_1^2 + \xi_2^2 + \xi_3^2 = -(H_{+1}H_{-1} + H_3^2 - H_3) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{4.3}$$

as the standard Casimir operator (3.20). Commutation relations for the Lie operator generated by the vectors (4.2) have the form

$$[L_{H_S}, L_{H_3}] = -sL_{H_S}\,, \quad (s = \pm 1); \qquad [L_{H_{+1}}, L_{H_{-1}}] = 2L_{H_3}. \tag{4.4}$$

The operators L_{H_S} are the creation and annihilation operators for tensor functions $\hat{T}^A_B$, which, in their turn, are the eigenfunctions of the operator L_{H_3}. Following [10] we can show that there is the set of tensor functions $T^{(l)A}_{(m)B}$ for which

$$\mathcal{L}_{H_3}T^{(l)A}_{(m)B} = mT^{(l)A}_{(m)B} \qquad (m = -l, -l+1, \ldots, l)\,, \tag{4.5}$$

$$\mathcal{L}_{H_S}T^{(l)A}_{(m)B} = \sqrt{l(l+1) - m(m+s)}T^{(l)A}_{(m)B}\,, \tag{4.6}$$

where l is the weight of the representation. Herewith the tensor eigenfunction equations (2.16) can be written in the form

$$-\mathcal{G}T^{(l)A}_{(m)B} = (\mathcal{L}_{H_{+1}}\mathcal{L}_{H_{-1}} + \mathcal{L}_{H_3}\mathcal{L}_{H_3} - \mathcal{L}_{H_3})T^{(l)A}_{(m)B} = l(l+1)T^{(l)A}_{(m)B}. \tag{4.7}$$

Suppose that we need to consider the spherical tensor symmetric harmonics of type $(2,0)$ and of weight l, which we shall denote T^l. With respect to the initial differential basis $\{e^r, \;\; dx^1 = d\theta, \;\; dx^2 = d\varphi\}$ they can be written in the form

$$T^l = T^{(l)}_{rr}e^r \otimes e^r + T^{(l)}_{ra}(e^r \otimes \; dx^a + dx^a \otimes e^r) + T^{(l)}_{ab}dx^a \otimes dx^b. \tag{4.8}$$

Note, that the covector $e^r = dr$ is invariant with respect to the rotation group. In order to split the system of equations (4.7) for the tensor (4.8), into the irreducible components, one should go over into the basis of one-forms on the surfaces of transitivity S^2 satisfying the condition (3.10). It turns out that the covectors

$$e^s = d\theta + \imath s \sin\theta d\varphi \qquad (s = \pm 1) \tag{4.9}$$

are required. Herewith, the condition of invariance of a split structure (3.7) for the Lie operators associated with the vectors (4.2) stipulates the relation

$$\mu^s_{s'} = \frac{s e^{\imath s' \varphi}}{\sin\theta}\,, \quad \mu^s_3 = 0 \qquad (s, s' = \pm 1). \tag{4.10}$$

By using (4.9) the relation (4.8) can be rewritten in the form

$$T^l = T^{(l)}_{rr} e^r \otimes e^r + T^{(l)}_{rs}(e^r \otimes e^s + e^s \otimes e^r) + T^{(l)}_{ss'} e^s \otimes e^{s'}. \tag{4.11}$$

where the sum on $s = \pm 1$ is implied. Then, if we suppose

$$T^{(l)}_{rr} = h_{rr} t^l\,; \quad T^{(l)}_{rs} = h_r t^l_s\,; \quad T^{(l)}_{rs} = h t^l_{s+s'}\,, \tag{4.12}$$

where $h_{rr},\ h_r,\ h$ are functions of r, then for the function $t^l_n \ \ (n = 0, ..., s, ..., s+s')$ we obtain

$$\left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \left(\frac{\partial^2}{\partial\varphi^2} - 2\imath\, n \cos\theta \frac{\partial}{\partial\varphi} - n^2 \right) + l(l+1) \right\} t^l_n = 0. \tag{4.13}$$

Owing to (4.5) we find

$$t^l_n = e^{\imath m\varphi} P^l_{nm}, \tag{4.14}$$

where the functions P^l_{nm} satisfy the ordinary differential equation following from (4.13):

$$\left\{ \frac{1}{\sin\theta} \frac{d}{d\theta} \sin\theta \frac{d}{d\theta} - \frac{m^2 - 2mn\cos\theta + n^2}{\sin^2\theta} + l(l+1) \right\} P^l_{nm} = 0. \tag{4.15}$$

The solutions of the obtained equations are the functions $P^l_{nm}(\cos\theta)$ [11], which appear in the theory of representation of the rotation group and are proportional to the Jacobi polynomials $P^{(\alpha,\beta)}_k$. Recurrent relations for P^l_{nm} follow from (4.5),(4.6).

V. THE GENERALIZED CASIMIR OPERATOR FOR THE BIANCHI TYPE G^3 II GROUP AND ITS TENSOR REPRESENTATIONS

Let us consider, briefly, the case of nonunitary representations for noncompact groups. We shall take as an example the three-parameter non-Abelian group acting on M^3 with the coordinates $\{x, y, z\}$, i.e. G^3 II according to the Bianchi classification [12]. The Lie algebra of this group is represented by the vectors

$$\xi_1 = \frac{\partial}{\partial x}\,; \qquad \xi_2 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\,; \qquad \xi_3 = \frac{\partial}{\partial z}\,; \qquad [\xi_1, \xi_2] = \xi_1\,. \tag{5.1}$$

If we, for simplicity, make a substitution $(x, y, z) \rightarrow (\upsilon = xe^{-y}, y, z)$, then

$$\xi_1 = e^{-y}\frac{\partial}{\partial \upsilon}\,; \qquad \xi_2 = \frac{\partial}{\partial y}\,; \qquad \xi_3 = \frac{\partial}{\partial z}\,; \qquad [\xi_1, \xi_2] = \xi_1\,. \tag{5.2}$$

Since the Cartan tensor (2.11) is degenerate in this case, then in order to find the non-degenerate metric on M^3 it is necessary that the Killing's equations (2.9) be solved at $C^1_{12} = 1$; $C^2_{12} = 0$; $C^3_{12} = 0$. This can easily be done by the constructing of the invariant vector basis $\{e_a\}$, which satisfies the equations $L_{\xi_a} e_b = 0$. The solution of the equation of the invariance can be expressed in the form

$$e_1 = \frac{\partial}{\partial \upsilon} = e^y \xi_1\,; \qquad e_2 = \partial y - \upsilon\frac{\partial}{\partial \upsilon} = \xi_2 - \upsilon e^y \xi_1\,; \qquad e_3 = \xi_3\,. \tag{5.3}$$

The required metric, according to (3.15), is written in the form

$$\begin{aligned} g^{-1} &= e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 \\ &= (1+\upsilon^2)e^{2y}\xi_1 \otimes \xi_1 - \upsilon e^y(\xi_1 \otimes \xi_2 + \xi_2 \otimes \xi_1) + \xi_2 \otimes \xi_2 + \xi_3 \otimes \xi_3\,. \end{aligned} \tag{5.4}$$

Hence, omitting the symbol of the tensor product, we obtain the generalized Casimir operator.

In the case being considered the finding of the tensor harmonics $\hat{T}^{(\lambda)A}_{(\mu\nu)B} = T^{(\lambda)A}_{(\mu\nu)B}\hat{e}^B_A$, where $T^{(\lambda)A}_{(\mu\nu)B} = h^A_B t^\lambda_{\mu\nu}$, reduces to the construction of the scalar harmonics $t^\lambda_{\mu\nu}$. The latter are the eigenfunctions of the operators

$$\xi_2 t^\lambda_{\mu\nu} = \frac{\partial}{\partial y} t^\lambda_{\mu\nu} = \mu t^\lambda_{\mu\nu}, \tag{5.5}$$

$$\xi_3 t^\lambda_{\mu\nu} = \frac{\partial}{\partial z} t^\lambda_{\mu\nu} = \nu t^\lambda_{\mu\nu}, \tag{5.6}$$

$$K t^\lambda_{\mu\nu} \equiv \left[(1+\upsilon^2)\frac{\partial^2}{\partial \upsilon^2} - 2\upsilon\frac{\partial^2}{\partial \upsilon \partial y} + \upsilon\frac{\partial}{\partial \upsilon} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] t^\lambda_{\mu\nu} = \lambda t^\lambda_{\mu\nu}. \tag{5.7}$$

where μ and ν are the eigenvalues of the operators ξ_2 and ξ_3 respectively. These relations are analogies of the equations (4.5), (4.7) for the group $SO(3)$. From the equations (5.5), (5.6) one finds

$$t^\lambda_{\mu\nu} = f^\lambda_{\mu\nu} e^{\mu y + \nu z}, \tag{5.8}$$

where $f^\lambda_{\mu\nu} = f^\lambda_{\mu\nu}(\upsilon)$ satisfies the ordinary differential equation that follows from (5.7)

$$\left[(1+\upsilon^2)\frac{d^2}{d\upsilon^2} + (1-2\mu)\upsilon\frac{d}{d\upsilon} + \mu^2 + \nu^2\right] f^\lambda_{\mu\nu} = \lambda f^\lambda_{\mu\nu}. \tag{5.9}$$

Since the index ν comes algebraically into (5.9) we can rewrite the equation (5.9) in the following form:

$$\left[(1+\upsilon^2)\frac{d^2}{d\upsilon^2}+(1-2\mu)\upsilon\frac{d}{d\upsilon}+\mu^2\right]\varphi^{\sigma}_{\mu}=\sigma\varphi^{\sigma}_{\mu}, \tag{5.10}$$

where we used the substitution: $f^{\lambda}_{\mu\nu}=A^{\lambda}_{\mu\nu}\varphi^{\sigma}_{\mu}, \quad A^{\lambda}_{\mu\nu}=const, \quad \sigma=\lambda-\nu^2$. The general solution of (5.10) can be written in the form

$$\begin{aligned}\varphi(\upsilon) &= A\,{}_2F_1([-\mu-\sqrt{2\mu^2-\sigma}]/2,[-\mu+\sqrt{2\mu^2-\sigma}]/2,1/2,-\upsilon^2)\\ &+\sqrt{\upsilon^2}B\,{}_2F_1([-1-\mu-\sqrt{2\mu^2-\sigma}]/2,[1-\mu+\sqrt{2\mu^2-\sigma}]/2,3/2,-\upsilon^2),\end{aligned} \tag{5.11}$$

where A and B are constant, and the hypergeometric function ${}_2F_1(a,b,c,z)$ is the solution of the following equation:

$$z(1-z)\frac{d^2F}{dz^2}+[c-(a+b+1)z]\frac{dF}{dz}-abF=0. \tag{5.12}$$

It is easy to show from the commutation relations (5.2) that

$$t^{\lambda}_{\mu-1,\nu}=\xi_1 t^{\lambda}_{\mu\nu}. \tag{5.13}$$

Hence it follows

$$\frac{d}{d\upsilon}f^{\lambda}_{\mu\nu}=f^{\lambda}_{\mu-1,\nu}. \tag{5.14}$$

Since the group $G^3\ II$ is noncompact, then the spectra of the operators (5.5),(5.7) are continuous. Here we shall consider, as one more example, point series of the representation of $G^3\ II$. For this purpose we shall take $\sigma=n^2$. Then for the case of $\mu=n$ from (5.10) we obtain

$$\varphi^{n^2}_{n} = C\int(1+\upsilon^2)^{n-1/2}\,d\upsilon, \tag{5.15}$$

where C is the constant of integration (an additive constant is omitted). Applying the relation (5.14) to (5.15) $n-m-1$ times one has

$$\varphi^{n^2}_{m} = C\frac{d^{n-m-1}}{d\upsilon^{n-m-1}}(1+\upsilon^2)^{n-1/2} \quad (m<n). \tag{5.16}$$

Then we immediately obtain the particular solutions of the equations (5.9) $f^{\lambda}_{\mu\nu}$ as $A^{\lambda}_{\mu\nu}\varphi^{\sigma}_{\mu}$ when $\sigma=n^2,\ \mu=n,\ \lambda=\nu^2+n^2$. That this relation is really the solution of the equation (5.9) one can easily verify by the differentiating of the equation (5.9) $n-m-1$ times under initial values $\sigma=\mu^2=n^2$.

In the conclusion we shall write, for instance, the tensor eigenfunctions of the generalized Casimir operator G and of the Lie operator $\mathcal{L}_{\xi_2}$ for the covector harmonic $A^{(\lambda)}_{(\mu\nu)}$. Using the invariant basis of the one-forms

$$e^1=d\upsilon+\upsilon\,dy=e^{-y}dx\ ; \quad e^2=dy\ ,\ e^3=dz$$

which is dual to the basis (5.3), we find

$$A^{(\lambda)}_{(\mu\nu)}=\left[a^{(\lambda)}_{(\mu\nu)1}e^{-y}dx+a^{(\lambda)}_{(\mu\nu)2}dy+a^{(\lambda)}_{(\mu\nu)3}dz\right]t^{\lambda}_{\mu\nu} \tag{5.17}$$

where

$$t^{n^2}_{m\nu} = e^{ym+z\nu}\frac{d^{n-m-1}}{d\upsilon^{n-m-1}}(1+\upsilon^2)^{n-1/2}; \quad \upsilon=xe^{-y}, \tag{5.18}$$

$a^{(\lambda)}_{(\mu)(\nu)1},\ a^{(\lambda)}_{(\mu)(\nu)2},\ a^{(\lambda)}_{(\mu)(\nu)3}$ are constant.

VI. CONCLUSION

In such a way, the possibilities of our method have been considered for the Bianchi type $G^3\ IX = SO(3)$ and $G^3\ II$ groups. Nonetheless it is evident that the present method, in fact, makes it possible to construct tensor representations of any continuous group G.

The question remains is whether it is possible to generalize this method not only for tensor but also for spinor fields. The notion of the Lie derivative of spinor fields was introduced by I. Kosmann (see [13]). In [14] by extending the spinor representation of the Lorentz group to the representation of the general linear group $GL(4)$ the spinor fields are considered in arbitrary frames, and thus there were defined the Lie derivatives of the spinor fields with respect to an arbitrary vector field. Recently a geometric definition of the Lie derivative for spinor fields, more general than Kosmann's one, has been proposed in [15]. When choosing special infinitesimal lift (namely, for Kosmann vector fields) their definition coincides with that given by I.Kosmann.

The essential property we used in our method is that the commutator of the Lie derivatives of tensor fields with respect to vector fields equals to the Lie derivatives of tensor fields with respect to the commutator of the vector fields. However for spinor fields, as it follows from [13], [14], this is not so. In this connection it seems alluring to give a new definition (if it is possible) of the Lie derivative for spinor fields that will satisfy this requirement.

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